New Lower Bound on Cycle Length for the $3n + 1$ Problem

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Collatz Function

Definition

The function $T : \mathbb{N} \to \mathbb{N}$ is given by

$$T(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{3n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Brief History

This problem is generally credited to Lother Collatz (hence the common name **Collatz Problem**). Having distributed the problem during the International Congress of Mathematicians in Cambridge.
Helmut Hasse is often attributed with this problem as well, and sometimes the iteration is referred to by the name **Hasse's algorithm**.
Another common name is the **Syracuse problem**.
Other names associated with the problem are S. Ulam, H. S. M. Coxeter, John Conway, and Sir Bryan Thwaites.

Behavior of function

We can look at how the function behaves at certain values:

$$T(5) = 8,\ T(12) = 6,\ T(49) = 74$$

or more interestingly we can look at the sequence of iterates of the function: $T^{(k)}(n)$:

$$(12, 6, 3, 5, 8, 4, 2, 1, 2, 1, \ldots)$$
$$(3, 5, 16, 8, 4, 2, 1, 2, 1, \ldots)$$
$$(17, 26, 13, 20, 10, 5, 8, 4, 2, 1, \ldots)$$
$$(256, 128, 64, 32, 16, 8, 4, 2, 1, \ldots)$$

$3n + 1$ problem

$3n + 1$ problem

Prove (or find a counter example) that every positive integer, n, eventually reaches the number 1, under iterations of the function $T(n)$.

Importance of the problem

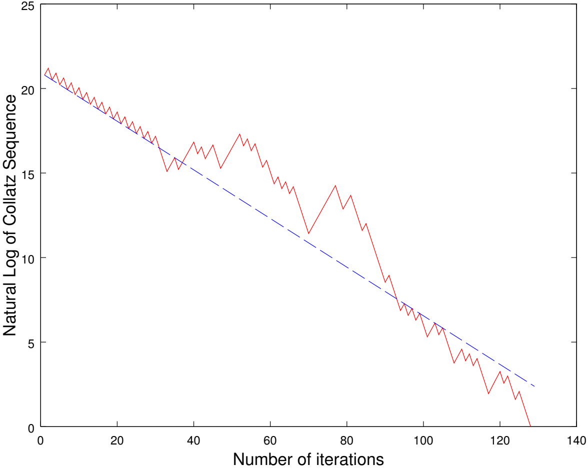
This problem has been analyzed from many different areas including:

1. Number Theory
2. Probability Theory
3. Dynamical Systems
4. Computer Science

This is a problem which is easy to immediately understand and play around with!

Heuristic Algorithm

One heuristic algorithm estimates the growth between successive odd iterations; suggesting that all sequences eventually begin decreasing.



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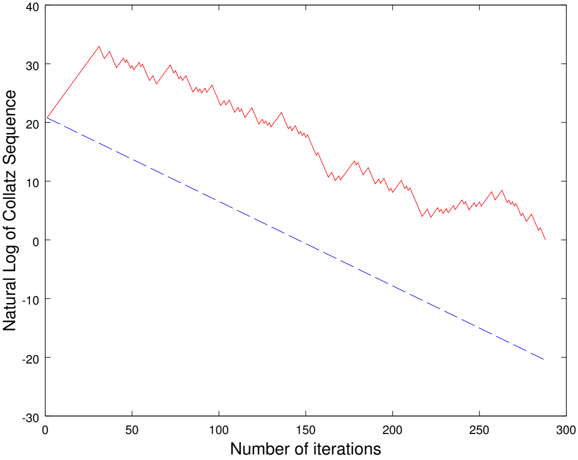

Supporting evidence

Tomás Oliverira e Silva (2011)

All integers up to $5 \cdot 2^{60} \approx 5.754 \times 10^{18}$ eventually reach one under The Collatz iteration.

C. J. Everett (1977)

The integers with finite total stopping time have natural density 1 in the natural numbers.

Major result

Theorem

Let Ω be a cycle of T and $\Omega_1 \subset \Omega$ be the odd elements of Ω then

$$\log_2(3 + M^{-1}) < \frac{|\Omega|}{|\Omega_1|} \leq \log_2(3 + m^{-1})$$

where $M = \max \Omega$ and $m = \min \Omega$.

Farey Pairs

Definition

Two fractions (any fractions, not necessarily convergents) $\frac{p}{q}$ and $\frac{p'}{q'}$ with p, q, p', q' non-negative integers and in reduced form are a **Farey pair** if $pq' - p'q = \pm 1$.

Example of Farey Pairs

Example

The following pairs form Farey pairs:

$$\frac{1}{4}, \frac{1}{3}$$
$$\frac{1}{3}, \frac{1}{2}$$
$$\frac{1}{2}, \frac{2}{3}$$
$$\frac{2}{3}, \frac{3}{4}$$

.

Farey Pair property

Lemma

Let $\frac{p}{q} < \frac{p'}{q'}$ form a Farey Pair. Then any intermediate fraction with $\frac{p}{q} < \frac{x}{y} < \frac{p'}{q'}$ $(y > 0)$. Then

$$\frac{x}{y} = \frac{ap + bp'}{aq + bq'}$$

with a, b positive integers. In particular $x \geq p + p'$ and $y \geq q + q'$.

Main Result

Theorem

Let Ω be a non-trivial cycle of T. Provided that $\min \Omega > 1.08 \times 2^{60}$ *we have*

$$|\Omega| = 630\,138\,877a + 10\,439\,860\,591b + 103\,768\,467\,013c$$

where a, b, c are non-negative integers with $b > 0$ and $ac = 0$. Specifically the smallest possible values for $|\Omega|$ are $10,439,860,591$*;* $11,069,999,488$*;* $11,700,138,385$*, etc.*

Proof

Using the notation $\frac{p_n}{q_n}$ is the rational approximation to $\log_2(3)$ with continued fraction with n terms. With (very) precise calculations we can observe

$$\frac{p_{22}}{q_{22}} < log_2(3) < \frac{p_{21}}{q_{21}} < log_2\left(3 + \frac{1}{5 \times 2^{60}}\right) < \frac{p_{19}}{q_{19}}$$

Using the notation $k = |\Omega|$ and $l = |\Omega_1|$ then the above inequalities (and a previous Theorem) give

$$\frac{k}{l} \in \left[\log_2(3), \log_2\left(3 + \frac{1}{5 \times 2^{60}}\right)\right]$$

Proof (cont.)

Re-writing our interval into three separate chunks gives three possible cases for where $\frac{k}{l}$ lands:

1. $\frac{k}{l} \in \left[\log_2(3), \frac{p_{21}}{q_{21}}\right) \subset \left(\frac{p_{22}}{q_{22}}, \frac{p_{21}}{q_{21}}\right)$
2. $\frac{k}{l} = \frac{p_{21}}{q_{21}}$
3. $\frac{k}{l} \in \left(\frac{p_{21}}{q_{21}}, \log_2\left(3 + \frac{1}{5 \times 2^{60}}\right)\right] \subset \left(\frac{p_{21}}{q_{21}}, \frac{p_{19}}{q_{19}}\right)$

Proof (cont.)

In the first case we have that

$$\frac{p_{22}}{q_{22}} < \frac{k}{l} < \frac{p_{21}}{q_{21}}$$

which form a Farey Pair. So by a previous result we have that $k = p_{21}b + p_{22}c$, with b and c positive integers.
In the second case we easily see that $k = p_{21}b$ where $b = 1$.
In the third case we get

$$\frac{p_{21}}{q_{21}} < \frac{k}{l} < \frac{p_{19}}{q_{19}}$$

which again form a Farey Pair. So here we get that $k = p_{19}a + p_{21}b$ with b and a positive integers.

Proof.

Since $\frac{k}{l}$ must fall in exactly one of these cases, we can write it all in one expression:

$$|\Omega| = 630\,138\,877a + 10\,439\,860\,591b + 103\,768\,467\,013c$$

(where we have replaced p_i with their numerical values).
Since we can't have both the a and the c case simultaneously it must be that at least one of them is 0, so $ac = 0$. And since b shows up in every case we have that $b > 0$. This completes the proof. $\square$

Final Thoughts

"Hopeless, absolutely hopeless"
"Mathematics is not yet ready for such problems"
–Paul Erdős
"Don't try to solve these problems"
–Richard Guy

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